Introduction to Hilbert C^*-modules

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Bounded module maps

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Definition 1.1

Let R be a ring. A right R-module M consists of an abelian group $(M, +)$ and an operation $\cdot : R \times M \to M$ such that for all $r, s \in R$ and $x, y \in M$, we have

$$(x+y)\cdot r = x\cdot r + y\cdot r, \tag{e 0.1}$$

$$x\cdot(r+s) = x\cdot r + x\cdot r, \tag{e 0.2}$$

$$(x\cdot(rs) = (x\cdot r)\cdot s, \tag{e 0.3}$$

$$x\cdot 1 = x. \tag{e 0.4}$$

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$H^\sharp$ is a right A-module with $\phi \cdot a(x) = a^*\phi(x)$ for all $x \in H$ and $a \in A$.

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Let H_1, H_2 be Hilbert A-modules. Denote by $B(H_1, H_2))$ the space of all bounded A-module maps from H_1 to H_2. Set $B(H) = B(H_1, H_2)$ if $H_1 = H_2$.

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Then $\{e_n\}$ forms an "orthonormal basis", i.e., for any $x \in H$, there are $a_n \in A$ such that $x = \sum_{n=1}^{\infty} e_n \cdot a_n$.
In general, H_A is not self-dual.

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Theorem

(Kasparov, 1980) Let A be a C^*-algebra and H be a countably generated Hilbert A-module. Then $H_A \oplus H \cong H_A$ (as Hilbert A-module).

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If H is a Hilbert A-module and $x, y \in H$, then $\theta_{x,y} : H \to H$ defined to be $\theta_{x,y}(z) = x\langle y, z\rangle$ for all $z \in H$.

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Let $\alpha \nearrow 1/2$. Note that $\|\langle y, y\rangle^{1-2\alpha}\| \to 1$ as $\alpha \nearrow 1$.

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Theorem

(Kasparov, 1980). There is an isometric isomorphism from $L(H)$ *onto* $M(K(H))$.

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Let $x \in H$. *Then* $x\langle x, x\rangle^{\beta_n} \to x$ *in norm* *if* $0 < \beta_n < 1$ *and* $\beta_n \to 0$,

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$$\|x - xa(a+1/n)^{-1}\|^2 = \|a(1 - a(a+1/n)^{-1})^2\| \qquad (e\,0.9)$$

$$= \|(a^{1/2} - a^{1+1/2}(a+1/n)^{-1})^2\| \to 0. \qquad (e\,0.10)$$

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To show that Φ is surjective, let $T_1 \in LM(K(H))$ and $x \in H$. By Lemma 1.10, choose $0 < \alpha < 1/2$ such that $3\alpha > 1$, $x = \xi\langle x, x\rangle^{\alpha}$ and $\langle \xi, \xi\rangle = \langle x, x\rangle^{1-2\alpha}$. Set $\zeta = \xi\langle x, x\rangle^{4\alpha-1}$ and, for any $b \in A$, $\eta = \xi\langle x, x\rangle^{3\alpha-1} bb^*$. Then, for any $y \in H$,

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It follows that $\psi(T_1)$ is a bounded module map and $\|\psi(T_1)\| \le \|T_1\|$.

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in norm. Therefore

$$\theta_{\psi(T_1)(x),y} = \lim_{n\to\infty} \theta_{(T_1\theta_{x,x}(x)(\langle x,x\rangle+1/n)^{-1},y} \qquad (e\,0.13)$$

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\theta_{\psi(T_1)(x),y} &= \lim_{n\to\infty} \theta_{(T_1\theta_{x,x}(x)(\langle x,x\rangle+1/n)^{-1},y} && (e\,0.13)\\
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&= \lim_{n\to\infty} \theta_{(T_1\theta_{w,w})(w)\langle x,x\rangle(\langle x,x\rangle+1/n)^{-1},y} && (e\,0.15)
\end{aligned}$$

To show the surjectivity of Φ, it suffices to show that $\Phi(\psi(T_1))(k) = T_1 k$ for all $k \in K(H)$. Since $\psi(T_1)\theta_{x,y} = \theta_{\psi(T_1)(x),y}$ for all $x, y \in H$, it is enough to show that $T_1\theta_{x,y} = \theta_{\psi(T_1)(x),y}$ for all $x, y \in H$. With notation above, let $w = \xi\langle x, x\rangle^{\beta}$ with $\beta = (3\alpha - 1)/3$. Then , for any $y \in H$,

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$$\begin{aligned}
\theta_{\psi(T_1)(x),y} &= \lim_{n\to\infty} \theta_{(T_1\theta_{x,x}(x)(\langle x,x\rangle+1/n)^{-1},y} && \text{(e 0.13)}\\
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&= \lim_{n\to\infty} \theta_{(T_1\theta_{w,w})(w)\langle x,x\rangle(\langle x,x\rangle+1/n)^{-1},y} && \text{(e 0.15)}\\
&= \theta_{(T_1\theta_{w,w})(w),y}. && \text{(e 0.16)}
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To show the surjectivity of Φ, it suffices to show that $\Phi(\psi(T_1))(k) = T_1 k$ for all $k \in K(H)$. Since $\psi(T_1)\theta_{x,y} = \theta_{\psi(T_1)(x),y}$ for all $x, y \in H$, it is enough to show that $T_1\theta_{x,y} = \theta_{\psi(T_1)(x),y}$ for all $x, y \in H$. With notation above, let $w = \xi\langle x, x\rangle^{\beta}$ with $\beta = (3\alpha - 1)/3$. Then , for any $y \in H$,

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\theta_{\psi(T_1)(x),y} &= \lim_{n\to\infty} \theta_{(T_1\theta_{x,x}(x)(\langle x,x\rangle+1/n)^{-1},y} && \text{(e 0.13)}\\
&= \lim_{n\to\infty} \theta_{(T_1\theta_{w,w})(\theta_{w,x}(x)(\langle x,x\rangle+1/n)^{-1},y} && \text{(e 0.14)}\\
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Therefore

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\theta_{\psi(T_1)(x),y} &= \lim_{n\to\infty} \theta_{(T_1\theta_{x,x}(x)(\langle x,x\rangle+1/n)^{-1},y} && \text{(e 0.17)}\\
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&= \lim_{n\to\infty} \theta_{(T_1\theta_{w,w})(w)\langle x,x\rangle(\langle x,x\rangle+1/n)^{-1},y} && \text{(e 0.19)}\\
&= \theta_{(T_1\theta_{w,w})(w),y}. && \text{(e 0.20)}
\end{aligned}$$

On the other hand,

$$T_1\theta_{x,y}$$

Therefore

$$
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On the other hand,

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T_1\theta_{x,y} = (T_1\theta_{w,w})\theta_{w,y} = \theta_{T_1\theta_{w,w})(w),y}.
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Therefore

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$$T_1\theta_{x,y} = (T_1\theta_{w,w})\theta_{w,y} = \theta_{T_1\theta_{w,w})(w),y}.$$

It follows that $T_1\theta_{x,y} = \theta_{\psi(T_1)(x),y}$ for all $x, y \in H$.

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It follows that $T_1\theta_{x,y} = \theta_{\psi(T_1)(x),y}$ for all $x, y \in H$. Consequently Φ is surjective.

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Therefore

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$$= \theta_{(T_1\theta_{w,w})(w),y}. \quad (e\,0.20)$$

On the other hand,

$$T_1\theta_{x,y} = (T_1\theta_{w,w})\theta_{w,y} = \theta_{T_1\theta_{w,w})(w),y}.$$

It follows that $T_1\theta_{x,y} = \theta_{\psi(T_1)(x),y}$ for all $x, y \in H$. Consequently Φ is surjective. Therefore Φ is an isometric isomorphism from the Banach algebra $B(H)$ onto the Banach algebra $LM(K(H))$.

Therefore

$$
\begin{aligned}
\theta_{\psi(T_1)(x),y} &= \lim_{n\to\infty} \theta_{(T_1\theta_{x,x}(x)(\langle x,x\rangle+1/n)^{-1},y} && \text{(e 0.17)}\\
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It follows that $T_1\theta_{x,y} = \theta_{\psi(T_1)(x),y}$ for all $x, y \in H$. Consequently Φ is surjective. Therefore Φ is an isometric isomorphism from the Banach algebra $B(H)$ onto the Banach algebra $LM(K(H))$. Note that $\Phi|_{K(H)} = \mathrm{id}_{K(H)}$.

Therefore

$$\begin{aligned}
\theta_{\psi(T_1)(x),y} &= \lim_{n\to\infty} \theta_{(T_1\theta_{x,x}(x)(\langle x,x\rangle+1/n)^{-1},y} && \text{(e 0.17)}\\
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It follows that $T_1\theta_{x,y} = \theta_{\psi(T_1)(x),y}$ for all $x, y \in H$. Consequently Φ is surjective. Therefore Φ is an isometric isomorphism from the Banach algebra $B(H)$ onto the Banach algebra $LM(K(H))$. Note that $\Phi|_{K(H)} = \mathrm{id}_{K(H)}$. It is also clear that $\Phi|_{L(H)}$ is the same map given by Kasparov.

Lemma *1.14*

Let H be a Hilbert A-module and $x \in H$. Suppose that $\phi \in H^\sharp$. Then there exists a sequence $\{\zeta_n\}$ in $\overline{xA}$ such that

$$\langle \zeta_n, \zeta_n \rangle \phi(x) \to \phi(x) \tag{e 0.21}$$

in norm as $k \to \infty$.

Proof: Let $U : \overline{xA} \to R = \overline{\langle x, x\rangle A}$ be the Hilbert A-module isomomorphism. Recall that $U(y)^* U(z) = \langle y, z \rangle$ for all $y, z \in \overline{xA}$. Choose $0 < \alpha_n < \alpha_{n+1} < 1$, $n \in \mathbb{N}$ such that $\alpha_n \nearrow 1/2$. By Proposition 1.10, there are $x_n \in \overline{xA}$ with $\|x_n\| \le \|\langle x, x\rangle^{1/2-\alpha}\|$ such that $x = x_n \langle x, x\rangle^{\alpha_n}$, $n \in \mathbb{N}$. Note that $\phi(x) = \phi(x_n)\langle x, x\rangle^{\alpha_n}$ for all $n \in \mathbb{N}$. Hence

$$\phi(x_n)\langle x, x\rangle^{1/2} \to \phi(x), \text{ as } n \to \infty, \tag{e 0.22}$$

in norm. Put $y_n = x_n \langle x, x\rangle^{1/n}$, $n \in \mathbb{N}$. Then $\phi(y_n) = \phi(x_n)\langle x, x\rangle^{1/n} \in R^*$.

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$$\phi(x_n)\langle x, x\rangle^{1/2} \to \phi(x), \text{ as } n \to \infty, \tag{e 0.23}$$

in norm. Put $y_n = x_n\langle x, x\rangle^{1/n}$, $n \in \mathbb{N}$. Then $\phi(y_n) = \phi(x_n)\langle x, x\rangle^{1/n} \in R^*$. Moreover

$$\phi(y_n)\langle x, x\rangle^{\alpha_n} = \phi(x_n)\langle x, x\rangle^{\alpha_n+1/n} \to \phi(x).$$

in norm. Put $v_n = \phi(y_n)^* \in R$. for all $n \in \mathbb{N}$. Let $z_n = U^{-1}(v_n)$. Then,

$$\langle z_n, x\langle x, x\rangle^{1/n}\rangle = v_n^*\langle x, x\rangle^{1/2+1/n} \to \phi(x).$$

By Lemma 1.12,

$$\langle z_n, x\rangle \to \phi(x). \tag{e 0.24}$$

By Lemma 1.10, for each $m \in \mathbb{N}$, we write $z_n = \xi_{n,m}\langle z_n, z_n\rangle^{\alpha_m}$ for some $\xi_{n,m} \in \overline{xA}$, where $\langle \xi_{n,m}, \xi_{n,m}\rangle = \langle z_n, z_n\rangle^{1-2\alpha_m}$, $n, m \in \mathbb{N}$. Let $w_{n,m} = \xi_{n,m}\langle z_n, z_n\rangle^{1/2m}$, $m \in \mathbb{N}$.

Put $v_n = \phi(y_n)^* \in R$. for all $n \in \mathbb{N}$. Let $z_n = U^{-1}(v_n)$. Then,

$$\langle z_n, x\langle x, x\rangle^{1/n}\rangle = v_n^*\langle x, x\rangle^{1/2+1/n} \to \phi(x).$$

By Lemma 1.12,

$$\langle z_n, x\rangle \to \phi(x). \tag{e 0.25}$$

By Lemma 1.10, for each $m \in \mathbb{N}$, we write $z_n = \xi_{n,m}\langle z_n, z_n\rangle^{\alpha_m}$ for some $\xi_{n,m} \in \overline{xA}$, where $\langle \xi_{n,m}, \xi_{n,m}\rangle = \langle z_n, z_n\rangle^{1-2\alpha_m}$, $n, m \in \mathbb{N}$. Let $w_{n,m} = \xi_{n,m}\langle z_n, z_n\rangle^{1/2m}$, $m \in \mathbb{N}$. Then, for fixed n,

$$z_n\langle w_{n,m}, w_{n,m}\rangle = \xi_{n,m}\langle z_n, z_n\rangle^{1-\alpha_m+1/m} = z_n\langle z_n, z_n\rangle^{1-2\alpha_m+1/m}.$$

Note that $\lim_{m\to\infty} 1 - 2\alpha_m + 1/m = 0$. By Lemma 1.12, $z_n\langle w_{n,m}, w_{n,m}\rangle \to z_n$ as $m \to \infty$. Therefore, there exists a subsequence $\{m(n)\}$ such that

$$\langle w_{n,m(n)}, w_{n,m(n)}\rangle\langle z_n, x\rangle \to \phi(x).$$

Hence

$$\lim_{n\to\infty}\langle w_{n,m(n)}, w_{n,m(n)}\rangle\phi(x) = \phi(x).$$

Put $\zeta_n = w_{n,m(n)}$. Then $\zeta_n \in \overline{xA}$ which meets the requirements.

Theorem 1.14 Let A be a C^*-algebra and H be a Hilbert A-module. Then there exists an isometric linear map Φ_1 from $B(H, H^\sharp)$ onto $QM(K(H))$. Moreover, the restriction of Φ_1 on $B(H)$ is the map described in Theorem 1.13.

Proof: Recall that $H^\sharp$ is a Banach A-module with $\phi \cdot a(x) = a^*\phi(x)$ for all $x \in H$ and $a \in A$. Denote by $F(H)$ the linear span of rank one module maps of the form $\theta_{x,y}$ $(x, y \in H)$. Recall also that $K(H)$ is the closure of $F(H)$. Define a map $\Phi_1 : B(H, H^\sharp) \to QM(K(H))$ by

$$\theta_{x',y'}\Phi_1(T)\theta_{x,y} = \theta_{x',y(T(x)(y'))} \text{ for all } T \in B(H, H^\sharp) \qquad (e\,0.26)$$

for any $x, y, x', y' \in H$. (recall that $T(x)(y') \in A$). Extend $\Phi_1(T)$ linearly to a map of the form $F(H) \times F(H) \to F(H)$.
Suppose that $\|x\| \le 1$ and $x = \xi\langle x, x\rangle^\alpha$ for some $0 < 1/3 < \alpha < 1/2$ as given by Lemma 1.10, where $\xi \in \overline{xA}$. Set $w = \xi\langle x, x\rangle^\delta$ for some $0 < \delta < 1/2$. In the next estimates, we will use the inequality $(T(w)(y'))^*(T(w)(y') \le \|T(w)\|^2\langle y', y'\rangle$.

For $y, z \in H$ and $a \in A$, we have

$$\begin{aligned}
&\|\theta_{x',y(T(x)(y'))}(z)\|^2 \\
&= \|\langle z,y\rangle(T(x)(y'))\langle x',x'\rangle(T(x)(y'))^*\langle y,z\rangle\| \\
&= \|\langle z,y\rangle\langle x,x\rangle^{\alpha-\delta}(T(w)(y'))\langle x',x'\rangle(T(w)(y'))^*\langle x,x\rangle^{\alpha-\delta}\langle y,z\rangle\| \\
&\le \|\langle x'x'\rangle^{1/2}(T(w)(y'))^*\|^2\|\|\langle x,x\rangle^{\alpha-\delta}\langle y,z\rangle\|^2 \\
&= \|\langle x'x'\rangle^{1/2}(T(w)(y'))^*(T(w)(y'))\langle x',x'\rangle^{1/2}\|\|\langle x,x\rangle^{\alpha-\delta}\langle y,z\rangle\|^2 \\
&\le \|T(w)\|^2\|\langle x',x'\rangle^{1/2}\langle y',y'\rangle\langle x',x'\rangle^{1/2}\|\|\langle x,x\rangle^{\alpha-\delta}\langle y,z\rangle\langle z,y\rangle\langle x,x\rangle^{\alpha-\delta}\| \\
&\le \|T(w)\|^2\|\langle x',x'\rangle^{1/2}\langle y',y'\rangle^{1/2}\|^2\|\langle x,x\rangle^{\alpha-\delta}\langle y,y\rangle^{1/2}\|^2\|z\|^2. \qquad (e\,0.27)
\end{aligned}$$

Let $\delta \to 0$. We obtain (with $\|x\| \le 1$)

$$\|\theta_{x',y(T(x)(y'))}(z)\| \le \|T\|\|\langle x',x'\rangle^{1/2}\langle y'y'\rangle^{1/2}\|\langle x,x\rangle^{\alpha}\langle y,y\rangle^{1/2}\|z\|.$$

Then, let $\alpha \to 1/2$. We further obtain

$$\|\theta_{x',y'}\Phi_1(T)\theta_{x,y}\| \le \|T\|\|\theta_{x',y'}\|\|\theta_{x,y}\| \qquad (e\,0.28)$$

for all $x, y, x', y' \in H$.

We then uniquely extend a map $\Phi_1(T) : K(H) \times K(H) \to K(H)$ which defines a quasi-multiplier of $K(H)$ and $\|\Phi_1(T)\| \le \|T\|$ for all $T \in B(H, H^\sharp)$. To see that $\|\Phi_1(T)\| = \|T\|$, we assume that $\|x\|, \|y\|, \|y'\| \le 1$. Put $\zeta = y(T(x)(y')$ and $\zeta = v\langle \zeta, \zeta\rangle^\alpha$ for some $1/3 < \alpha < 1/2$, where $v \in \overline{\zeta A}$ and $\langle v, v\rangle = \langle \zeta, \zeta\rangle^{1-2\alpha}$. For $\eta > 0$, choose $x' = v\langle \zeta, \zeta\rangle^\eta / \|\|v\langle \zeta, \zeta\rangle^\eta\|$. Then $\|x'\| \le 1$. Note that

$$\begin{aligned} \langle x', x'\rangle &= \langle v, v\rangle\langle \zeta, \zeta\rangle^\eta / \|\|v\langle \zeta, \zeta\rangle^\eta\| \\ &= \frac{\langle yT(x)(y'), yT(x)(y')\rangle^{1-2\alpha+\eta}}{\|\langle yT(x)(y'), yT(x)(y')\rangle^{1-2\alpha+\eta}\|} \qquad (e\,0.29) \\ &= \frac{((T(x)(y'))^*\langle y, y\rangle(T(x)(y')))^{1-2\alpha+\eta}}{\|((T(x)(y'))^*\langle y, y\rangle(T(x)(y')))^{1-2\alpha+\eta}\|}. \qquad (e\,0.30) \end{aligned}$$

It follows that

$$\begin{aligned} \langle x', x'\rangle(T(x)(y'))^*\langle y, y\rangle(T(x)(y')) \to \qquad (e\,0.31) \\ (T(x)(y'))^*\langle y, y\rangle(T(x)(y')) \qquad (e\,0.32) \end{aligned}$$

as $\eta \to 0$ and $\alpha \to 1/2$.

We have, by (e 0.31), when $\delta \to 0$ and $\alpha \to 1/2$,

$$\begin{aligned} \|\theta_{x',y'}\Phi_1(T)\theta_{x,y}\| &= \|\theta_{x',yT(x)(y')}\| \\ &= \|\langle x',x'\rangle^{1/2}((T(x)(y'))^*\langle y,y\rangle(T(x)(y')))^{1/2}\| \\ &\to \|\langle y,y\rangle^{1/2}T(x)(y')\| \qquad \text{(e 0.33)} \end{aligned}$$

For any $\epsilon > 0$, there are $x, y' \in H$ with $\|x\| \le 1$ and $\|y'\| \le 1$ such that

$$\|T(x)(y')\| > \|T\| - \epsilon/2. \qquad \text{(e 0.34)}$$

Then, by (e 0.33), for sufficiently small δ and α close to $1/2$, by applying Lemma 1.10 and by choosing a y in the unit ball of H properly

$$\|\theta_{x',y'}\Phi_1(T)\theta_{x,y}\| \ge \|T\| - \epsilon. \qquad \text{(e 0.35)}$$

This implies that $\|\Phi_1(T)\| = \|T\|$. So Φ_1 is an isometry from $B(H, H^\sharp)$. Next we will show that Φ_1 is surjective. Let $T_1 \in QM(K(H))$. For any $k \in K(H)$, we have $k \cdot T_1 \in LM(K(H))$. For $x, y \in H$, write $y = \xi_1\langle y, y\rangle^\alpha$ (for some $1/3 < \alpha < 1/2$) with $\langle \xi_1, \xi_1\rangle = \langle y, y\rangle^{1-2\alpha}$ and define $\zeta_1 = \xi_1\langle y, y\rangle^{2\alpha - 1/2}$.

We verify that, for any $u \in H$,

$$\begin{aligned} \theta_{\zeta_1,\zeta_1}\theta_{\xi_1,\xi_1}(u) &= \zeta_1\langle\zeta_1,\xi_1\rangle\langle\xi_1,u\rangle = \xi_1\langle y,y\rangle^{4\alpha-1}\langle y,y\rangle^{1-2\alpha}\langle\xi_1,u\rangle \\ &= \xi_1\langle y,y\rangle^{2\alpha}\langle\xi_1,u\rangle = y\langle y,u\rangle. \end{aligned} \tag{e 0.36}$$

In other words, $\theta_{\zeta_1,\zeta_1}\theta_{\xi_1,\xi_1} = \theta_{y,y}$.
Let ψ be the same notation used in the proof of Theorem 1.13. Define

$$(\psi_1(T_1))(x)(y) = \lim_{n\to\infty}\langle\psi(\theta_{y,y}T_1)(x),y\rangle(\langle y,y\rangle+1/n)^{-1}$$

$$= \lim_{n\to\infty}\langle\psi(\theta_{\zeta_1,\zeta_1}\theta_{\xi_1,\xi_1}T_1)(x),y\rangle(\langle y,y\rangle+1/n)^{-1} \tag{e 0.37}$$

$$\lim_{n\to\infty}\langle\psi(\theta_{\zeta_1,\zeta_1})\psi(\theta_{\xi_1,\xi_1}T_1)(x),y\rangle(\langle y,y\rangle+1/n)^{-1} \tag{e 0.38}$$

$$= \lim_{n\to\infty}\langle\psi(\theta_{\xi_1,\xi_1}T_1)(x),\theta_{\zeta_1,\zeta_1}(y)\rangle(\langle y,y\rangle+1/n)^{-1} \tag{e 0.39}$$

$$= \lim_{n\to\infty}\langle\psi(\theta_{\xi_1,\xi_1}T_1)(x),\zeta_1\rangle\langle\zeta_1,y\rangle(\langle y,y\rangle+1/n)^{-1} \tag{e 0.40}$$

$$= \lim_{n\to\infty}\langle\psi(\theta_{\xi_1,\xi_1}T_1)(x),\xi_1\rangle\langle y,y\rangle^{3\alpha}(\langle y,y\rangle+1/n)^{-1} \tag{e 0.41}$$

$$= \langle\psi(\theta_{\xi_1,\xi_1}T_1)(x),\xi_1\rangle\langle y,y\rangle^{3\alpha-1} \tag{e 0.42}$$

(converges in norm as $3\alpha - 1 > 0$).

This shows that, for any $x \in H$, $\psi_1(T_1)(x)$ is a linear map from H to A. If we choose $\|y\| = 1$, then, by (**??**), we have

$$\|(\psi_1(T_1))(x)(y)\| = \|\langle \psi(\theta_{\xi_1,\xi_1} T_1)(x), \xi_1\rangle \langle y, y\rangle^{3\alpha-1}\| \tag{e 0.43}$$

$$\leq \|\psi(\theta_{\xi,\xi} T_1)\| \|x\| \leq \|\theta_{\xi,\xi} T_1\| \|x\| \leq \|T_1\| \|x\|. \tag{e 0.44}$$

This shows that $\psi_1(T_1)$ is a bounded linear map from H to $H^\sharp$. As in the proof of Theorem 1.13, in fact, it is a bounded module map in $B(H, H^\sharp)$. To show that Φ_1 is surjective, we need to show that $\Phi_1(\psi_1(T_1)) = T_1$. It then suffices to show that $\theta_{x',x'} T_1 \theta_{x,y} = \theta_{x,y(\psi_1(T_1)(x)(y'))}$ for $T_1 \in QM(K(H))$ and $x, y, x', y' \in H$. With $1/3 < \alpha < 1/2$, we keep write $x = \xi\langle x, x\rangle^\alpha$ as above, and $y' = \xi'\langle y', y'\rangle^\alpha$ with $\langle \xi', \xi'\rangle = \langle y', y'\rangle^{1-2\alpha}$. Set $w_1 = \xi\langle x, x\rangle^{\alpha-1/3}$ and $w_2 = \xi'\langle y', y'\rangle^{\alpha-1/2}$. From the proof of Theorem 1.13 (see (e 0.12)) we know that, for $S \in LM(K(H))$, $\psi(S)(x) = S\theta_{w_1,w_1}(w_1)$. Hence

$$\theta_{x',y(\psi_1(T_1)(x)(y'))} = \lim_{n\to\infty} \theta_{x',y\langle \psi(\theta_{y',y'} T_1)(x), y'\rangle [\langle y',y'\rangle + 1/n]^{-1}} \tag{e 0.45}$$

$$= \lim_{n\to\infty} \theta_{x',y\langle \psi(\theta_{w,w} T_1)(x), w\rangle \langle y',y\rangle [\langle y',y'\rangle + 1/n]^{-1}} \tag{e 0.46}$$

Hence

$$\begin{aligned}
\theta_{x',y(\psi_1(T_1)(x)(y'))} &= \lim_{n\to\infty} \theta_{x',y\langle\psi(\theta_{y',y'}T_1)(x),y'\rangle[\langle y',y'\rangle+1/n]^{-1}} \\
&= \lim_{n\to\infty} \theta_{x',y\langle\psi(\theta_{w,w}T_1)(x),w\rangle\langle y',y\rangle[\langle y',y'\rangle+1/n]^{-1}} \\
&= \theta_{x',y\langle\psi(\theta_{w,w}T_1)(x),w\rangle}. \qquad (e\,0.47)
\end{aligned}$$

On the other hand,

$$\theta_{x',y'}T\theta_{x,y} = \theta_{x',w_2}\theta_{w_2,w_2}T_1\theta_{w_1,w_1}\theta_{w_1,y} = \theta_{x',y\langle\theta_{w_2,w_2}T_1\theta_{w_1,w_1}(w_1),w_2\rangle}.$$

Thus $\theta_{x'y'}T_1\theta_{x,y} = \theta_{x',y(\psi_1(T_1)(x)(y'))}$. It follows $\Phi_1(\psi_1(T_1)) = T_1$ and Φ_1 is surjective. Note also that the restriction of Φ_1 on $L(H)$ is Φ defined in Theorem 1.13.

Examples

Let A be a unital C^*-algebra which has a sequence of positive elements $\{d_n\}$ such that $\|d_n\| = 1$ and $d_i d_j = 0$ if $i \neq j$.

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Q: When are $B(H) = L(H)$? (or $M(B) = LM(B)$?)

Proposition

Let A be a separable C^*-algebra and $H = l^2(A)$. Then $B(H) = L(H)$ if and only if A is separable and dual.

A C^*-algebra C is a dual C^*-algebra, if $C \cong \oplus_{n=1}^{\infty} C_n$, where $C_n \cong K$, or $C_n = M_{r(n)}$ for some $r(n) \in \mathbb{N}$, or $C_n = \{0\}$.

Example:

(1) If A is unital, then $A = M(A) = LM(A) = QM(A)$.

(2) Let A be a commutative C^*-algebra. Then $M(A) = LM(A) = QM(A)$.

(3) If $M(C) = LM(C)$, and $B = M_r(C)$, $LM(B) = M(B)$.

(4) Let A be a non-unital but σ-unital simple C^*-algebra. Then $M(A) = LM(A)$ if and only if A is elementary.

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such that I_{i+1}/I_i is a dual C^*-algebra, $i = 0, 1, ..., n-1$.